Exotic projective structures and quasifuchsian spaces II

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Abstract

Let $P(S)$ be the space of projective structures on a closed surface S of genus $g > 1$ and let $Q(S)$ be the subset of $P(S)$ of projective structures with quasifuchsian holonomy. It is known that $Q(S)$ consists of infinitely many connected components. In this paper, we will show that the closure of any exotic component of $Q(S)$ is not a topological manifold with boundary and that any two components of $Q(S)$ have intersecting closures.

1 Introduction

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a (G, X)-structure, where X is the Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphisms of $\widehat{\mathbb{C}}$. We consider the space of marked projective structures $P(S)$ on S and its open subset $Q(S)$ of projective structures with quasifuchsian holonomy. It is known that $Q(S)$ consists of infinitely many connected components. The aim of this paper is to study how components of $Q(S)$ lies in $P(S)$, especially how these components bump or self-bump. Here we say that components $\mathcal{Q}$, $\mathcal{Q}'$ of $Q(S)$ *bump* if they have intersecting closures and that a component $\mathcal{Q}$ *self-bumps* if there is a point $Y \in \partial\mathcal{Q}$ such that for any sufficiently small neighborhood U of Y the intersection $U \cap \mathcal{Q}$ is disconnected.

Now let $R(S)$ be the set of conjugacy classes of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and $\mathcal{QF} \subset R(S)$ the subset of faithful representations with quasifuchsian images. It is a result of Hejhal [He] that the holonomy map $hol : P(S) \to R(S)$, taking a projective structure to its holonomy representation, is a local homeomorphism. Therefore, studying how $Q(S) = hol^{-1}(\mathcal{QF})$ lies in $P(S)$ is closely related to studying how the quasifuchsian space $\mathcal{QF}$ lies in the representation space $R(S)$. It is known by Goldman [Go] that the set of connected components of $Q(S)$ are classified by the set $\mathcal{ML}_{\mathbb{N}} = \mathcal{ML}_{\mathbb{N}}(S)$ of integral points of measured laminations; see §2.3–2.4. We denote by $\mathcal{Q}_\lambda$ the component of $Q(S)$ associated to $\lambda \in \mathcal{ML}_{\mathbb{N}}$, where $\mathcal{Q}_0$ is the component of standard projective structures. Here we say that $Y \in Q(S)$ is *standard* if its developing map is injective; otherwise it is *exotic*. One of the most important result on the distribution of components of $Q(S)$ in $P(S)$ is obtained by McMullen; see Appendix A in [Mc]:

Theorem 1.1 (McMullen). *There exists a sequence of exotic projective structures which converges to a point in the relative boundary $\partial\mathcal{Q}_0 = \overline{\mathcal{Q}_0} - \mathcal{Q}_0$ of the standard component $\mathcal{Q}_0$ in $P(S)$.*

In fact, given non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, McMullen obtained in [Mc] a both-sides infinite convergent sequence

$$Y_n \to Y_\infty \in \partial\mathcal{Q}_0 \quad (|n| \to \infty)$$

associated to λ by using the method developed by Anderson and Canary [AC]. In addition, we showed in [It1] that the projective structures Y_n above are contained in the exotic component $\mathcal{Q}_\lambda$ for all large $|n|$, and thus obtained the following:

Theorem 1.2 (Theorem A in [It1]). *For any non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda} \neq \emptyset$.*

Now let μ be a non-zero element of $\mathcal{ML}_{\mathbb{N}}$ which has no parallel component in common with λ. Then one can obtain the grafting $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ of Y_∞ along μ (see §2.4), which lies in the boundary of $\mathcal{Q}_\mu$. Since the the map *hol* is a local homeomorphism, there is a both-sides infinite convergent sequence

$$Z_n \to Z_\infty \in \partial\mathcal{Q}_\mu \quad (|n| \to \infty)$$

which satisfies $hol(Z_n) = hol(Y_n)$ for all large $|n|$. Although the sequences $\{Y_n\}_{n\gg 0}$ and $\{Y_n\}_{n\ll 0}$ are contained in the same component $\mathcal{Q}_\lambda$, the sequences $\{Z_n\}_{n\gg 0}$ and $\{Z_n\}_{n\ll 0}$ are not necessarily contained in the same component. In fact, the main theorem in this paper (Theorem 1.3 below) states that these sequences are contained in components corresponding to elements $(\lambda,\mu)_\sharp$, $(\lambda,\mu)_\flat$ in $\mathcal{ML}_{\mathbb{N}}$, respectively, which are defined in §2.5. See Figure 1. We just remark here that $(\lambda,\mu)_\sharp \neq (\lambda,\mu)_\flat$ if and only if $i(\lambda,\mu) \neq 0$, where $i(\lambda,\mu)$ is the geometric intersection number of λ and μ.

Theorem 1.3. *In the same notation as above, we have $\{Z_n\}_{n\gg 0} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n\ll 0} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$.*

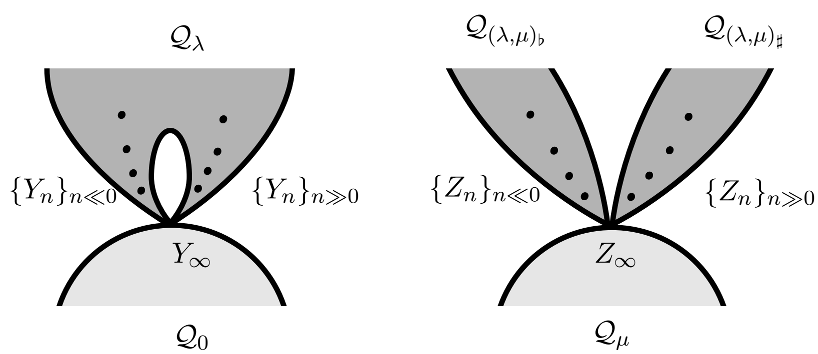

Figure 1: Sequences $\{Y_n\}_{|n|\gg 0}$ and $\{Z_n\}_{|n|\gg 0}$.

As consequences of Theorem 1.3, we obtain the following results on the distribution of components of $Q(S)$; see Theorems 4.1, 4.3 and 4.4.

Corollary 1.4. *(1) For any non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, $\mathcal{Q}_\lambda$ self-bumps.*

(2) For any $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}$, we have $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$.

(3) For any non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the holonomy map $hol : P(S) \to R(S)$ is not injective on $\overline{\mathcal{Q}_\lambda}$, although it is injective on $\mathcal{Q}_\lambda$.

Figure 2 is a computer generated figure by Komori, Sugawa, Wada, and Yamashita (cf. [KS] and [KSYW]), which is a part of a complex one-dimensional slice of $P(S)$ for a punctured-torus S. All the projective structures in this slice have the same underlying complex structure. The region in white is a slice of $Q(S)$: the inner disk is a Bers slice (a slice of the standard component $\mathcal{Q}_0$) and the outer part is a slice of an exotic component. This figure seems to illustrate the phenomenon stated in Theorem 1.3 and Corollary 1.4 (1).

Figure 2: A part of a slice of $Q(S)$ in $P(S)$ (white part) for a punctured-torus S.

We now consider associated results on the quasifuchsian space $\mathcal{QF}$ in $R(S)$. Let ρ_n and ρ_∞ denote holonomy representations of Y_n and Y_∞ respectively. Theorem 1.1 then implies that $\mathcal{QF}$ self-bumps at ρ_∞ in $R(S)$; namely, for any neighborhood U of ρ_∞ in $R(S)$, there exits a neighborhood $V \subset U$ of ρ_∞ such that $V \cap \mathcal{QF}$ is disconnected; see [Mc]. As a direct consequence of Theorem 1.3 in the case of $i(\lambda, \mu) \neq 0$, we can refine this statement as follows:

Theorem 1.5. *For any neighborhood U of ρ_∞ in $R(S)$, there exits a neighborhood $V \subset U$ of ρ_∞ such that for all large enough $n > 0$, ρ_n and ρ_{-n} are contained in distinct components of $V \cap \mathcal{QF}$.*

Remark. The same result is obtained independently by Bromberg and Holt (oral communication with Bromberg; see also Holt [Ho]).

We mention here the relationship between the topics in this paper and the topology of deformation spaces of Kleinian groups. Let Γ be a finitely generated Kleinian group with non-trivial space $AH(\Gamma)$ of conjugacy classes of discrete faithful representations $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$. In general, the interior of $AH(\Gamma)$ consists of finitely many components. It was first shown by Anderson and Canary [AC] that for some Kleinian

group those components bumps. This result is generalized by Anderson, Canary and Metallurgy [ACM]. In our setting, Γ is a quasifuchsian group isomorphic to $\pi_1(S)$, and the quasifuchsian space $\mathcal{QF}$ is the interior of the space $AH(\Gamma)$ which consists of exactly one connected component. Also in this case, the idea of Anderson and Canary can be applied to show that $\mathcal{QF}$ self-bumps by using projective structures; see Theorem 1.1 due to McMullen. In fact, by lifting $\mathcal{QF} \subset R(S)$ to $Q(S) \subset P(S)$ via the holonomy map *hol*, we can discuss the bumping and self-bumping of components of $Q(S)$. After McMullen, Bromberg and Holt [BH] characterized the self-bumping of components of the interior of $AH(\Gamma)$ for general Kleinian groups Γ without using projective structures. Our results, especially (1) and (2) in Corollary 1.4, can be viewed as the projective structure analogues of the works in [AC], [ACM] and [BH]. We refer the reader to [Ca] for further information on the bumping and self-bumping of deformation spaces of Kleinian groups.

This paper is organized as follows: In section 2, we provide definitions and basic properties of the spaces and maps with which we are concerned. We devote section 3 to the proof of Theorem 1.3. The idea of the proof can be found at the beginning of this section. Corollaries of Theorem 1.3 are obtained in section 4.

2 Preliminaries

We refer the reader to [It1] for more information on some topics in this section. See also an exposition [It2].

2.1 Kleinian groups

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, which acts on the hyperbolic space $\mathbb{H}^3$ as isometries, and on the sphere at infinity $S^2_\infty = \widehat{\mathbb{C}}$ as conformal automorphisms. The *region of discontinuity* $\Omega(\Gamma)$ is the largest open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously, and the *limit set* $\Lambda(\Gamma)$ of Γ is its complement $\widehat{\mathbb{C}} - \Omega(\Gamma)$. The quotient manifold $N_\Gamma = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is called the *Kleinian manifold* of Γ. A *quasifuchsian group* Γ is a Kleinian group whose limit set $\Lambda(\Gamma)$ is a Jordan curve and which contains no element interchanging the two components of $\Omega(\Gamma)$. A *b-group* Γ is a Kleinian group which has exactly one simply connected invariant component of $\Omega(\Gamma)$, which is denoted by $\Omega_0(\Gamma)$.

Let $R(S)$ denote the space of all conjugacy classes $[\rho]$ of representations of $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(\pi_1(S))$ is non-abelian. For simplicity, we denote $[\rho]$ by ρ if there is no confusion. The space $R(S)$ is endowed with algebraic topology and is known to be a $(6g-6)$-dimensional complex manifold (see Theorem 4.21 in [MT] for example). *Quasifuchsian space* $\mathcal{QF} = \mathcal{QF}(S)$ is the subset of $R(S)$ consisting of faithful representations whose images are quasifuchsian groups. Then it is known by Bers [Be1], Marden [Ma] and Sullivan [Su] that $\mathcal{QF}$ is connected, contractible and open in $R(S)$.

2.2 Space of projective structures

A projective structure on S is a (G, X)-structure where X is the Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphism of $\widehat{\mathbb{C}}$. Let $P(S)$ denote the space of marked projective structures on S, or the space of equivalence classes of pairs (g, Y) of a projective surface Y and an orientation preserving homeomorphism $g : S \to Y$. Two pairs (g_1, Y_1) and (g_2, Y_2) are said to be equivalent if there is an isomorphism $h : Y_1 \to Y_2$ of projective structures such that $h \circ g_1$ is isotopic to g_2. The equivalence class of (g, Y) is simply denoted by Y.

A projective structure $Y \in P(S)$ has an underlying conformal structure $\pi(Y) \in T(S)$, where $T(S)$ is the Teichmüller space for S. The space $P(S)$ is equipped with a structure of a complex $(6g-6)$-dimensional holomorphic affine bundle over $T(S)$ with the projection $\pi : P(S) \to T(S)$.

A projective structure Y on S can be lifted to that $\widetilde{Y}$ on $\widetilde{S}$, where $\widetilde{S} \to S$ is the universal cover on which $\pi_1(S)$ acts as a covering group. Since $\widetilde{Y}$ is simply connected, we obtain a developing map $f_Y : \widetilde{Y} \to \widehat{\mathbb{C}}$ by continuing the charts analytically. In addition, the developing map induces a holonomy representation $\rho_Y : \pi_1(S) \cong \pi_1(Y) \to \mathrm{PSL}_2(\mathbb{C})$ which satisfies $f_Y \circ \gamma = \rho_Y(\gamma) \circ f_Y$ for every $\gamma \in \pi_1(S)$. We remark that the pair (f_Y, ρ_Y) is determined uniquely up to the canonical action of $\mathrm{PSL}_2(\mathbb{C})$. We now define the *holonomy map*

$$hol : P(S) \to R(S)$$

by $Y \mapsto [\rho_Y]$. Then Hejhal [He] showed that the map *hol* is a local homeomorphism and Earle [Ea] and Hubbard [Hu] independently showed that the map is holomorphic:

Theorem 2.1 (Hejhal, Earle and Hubbard). *The holonomy map $hol : P(S) \to R(S)$ is a holomorphic local homeomorphism.*

We denote by $Q(S) = hol^{-1}(\mathcal{QF})$ the set of projective structures with quasi-fuchsian holonomy. An element of $Q(S)$ is said to be *standard* if its developing map is injective; otherwise it is *exotic*. Let $\mathcal{Q}_0 \subset Q(S)$ denote the set of all standard projective structures. Then the map $hol|_{\mathcal{Q}_0} : \mathcal{Q}_0 \to \mathcal{QF}$ is a biholomorphism, which takes the Bers' embedded image of the Theichmüller space $T(S)$ in a fiber to a Bers slice (cf. [Be2]).

2.3 Integral points of measured laminations

Let $\mathcal{S} = \mathcal{S}(S)$ denote the set of homotopy classes of non-trivial simple closed curves on S. By abuse of the notation, we also denote a representative of $c \in \mathcal{S}$ by c. Let $\mathcal{ML}_{\mathbb{N}} = \mathcal{ML}_{\mathbb{N}}(S)$ denote the set of integral points of measured laminations on S, or the set of formal summation $\sum_{i=1}^{l} k_i c_i$ of mutually distinct, disjoint elements $c_i \in \mathcal{S}$ with positive integer k_i weights. We regard $\mathcal{S} \subset \mathcal{ML}_{\mathbb{N}}$. The "zero-lamination" 0 is also contained in $\mathcal{ML}_{\mathbb{N}}$. A *realization* $\widehat{\lambda}$ of $\lambda = \sum_{i=1}^{l} k_i c_i \in \mathcal{ML}_{\mathbb{N}}$ is a disjoint union of simple closed curves which realize each weighted simple closed curve $k_i c_i$ by k_i-parallel disjoint simple closed curves homotopic to c_i.

For c, $d \in \mathcal{S}$, the geometric intersection number $i(c,d)$ is the minimum number of points in which the representations of c and d must intersect. Note that $i(c,c)=0$ for any $c \in \mathcal{S}$. We naturally extend the definition of the geometric intersection number for elements of $\mathcal{ML}_{\mathbb{N}}$ by linearity. If $i(\lambda,\mu)=0$ for λ, $\mu \in \mathcal{ML}_{\mathbb{N}}$, we can define $m\lambda + n\mu \in \mathcal{ML}_{\mathbb{N}}$ for $m, n \in \mathbb{N}$.

2.4 Grafting

Let $Y \in P(S)$ and $\widetilde{Y} \to Y$ the universal cover. A simple closed curve $c \in \mathcal{S}$ is called *admissible* on Y if a connected component $\widetilde{c} \subset \widetilde{Y}$ of the preimage of $c \subset Y$ is mapped injectively by the developing map $f_Y : \widetilde{Y} \to \widehat{\mathbb{C}}$ onto its image $f_Y(\widetilde{c})$ and if the holonomy image $\rho_Y(c)$ fixing $f_Y(\widetilde{c})$ is loxodromic. We say that $\lambda \in \mathcal{ML}_{\mathbb{N}}$ is admissible on Y if every component of the support of λ is admissible.

Suppose that $c \in \mathcal{S}$ is admissible on $Y \in P(S)$. Let

$$A_c = (\widehat{\mathbb{C}} - f_Y(\widetilde{c}))/\langle \rho_Y(c) \rangle$$

be the quotient annulus with its induced projective structure. Then the *grafting* $\mathrm{Gr}_c(Y)$ is the projective surface obtained by cutting Y along c and inserting the annulus A_c without twisting. Similarly, we can define the grafting $\mathrm{Gr}_\lambda(Y)$ for admissible $\lambda \in \mathcal{ML}_{\mathbb{N}}$ by linearity. The basic fact is that the grafting operation does not change the holonomy representation; i.e. $hol(\mathrm{Gr}_\lambda(Y)) = hol(Y)$ holds for every admissible $\lambda \in \mathcal{ML}_{\mathbb{N}}$. Since the map $hol|_{\mathcal{Q}_0} : \mathcal{Q}_0 \to \mathcal{QF}$ is a biholomorphism and the grafting map

$$\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to P(S)$$

for $\lambda \in \mathcal{ML}_{\mathbb{N}}$ satisfies $hol \circ \mathrm{Gr}_\lambda = hol$, the map Gr_λ turns out to be a biholomorphism from $\mathcal{Q}_0$ onto its image $\mathcal{Q}_\lambda = \mathrm{Gr}_\lambda(\mathcal{Q}_0)$. From Goldman's grafting theorem [Go] below, we obtain the decomposition of $Q(S)$ into its connected components;

$$Q(S) = \bigsqcup_{\lambda \in \mathcal{ML}_{\mathbb{N}}} \mathcal{Q}_\lambda.$$

Theorem 2.2 (Goldman). *Suppose that $Y \in \mathcal{Q}_0$. Then every projective structure with holonomy $hol(Y)$ is obtained by grafting of Y along some $\lambda \in \mathcal{ML}_{\mathbb{N}}$.*

2.5 Operations on $\mathcal{ML}_{\mathbb{N}}$

We now introduce two operations

$$(\cdot,\cdot)_\sharp,\ (\cdot,\cdot)_\flat : \mathcal{ML}_{\mathbb{N}} \times \mathcal{ML}_{\mathbb{N}} \to \mathcal{ML}_{\mathbb{N}},$$

which is closely observed by Luo in [Lu]. For λ, $\mu \in \mathcal{ML}_{\mathbb{N}}$, the elements $(\lambda,\mu)_\sharp$ and $(\lambda,\mu)_\flat$ in $\mathcal{ML}_{\mathbb{N}}$ are obtained as follows: Let $\widehat{\lambda}$, $\widehat{\mu}$ be realizations of λ, μ whose geometric intersection number is minimal. Next resolve all intersecting points in $\widehat{\lambda} \cup \widehat{\mu}$ as in Figure 3. Then $(\lambda,\mu)_\sharp$ and $(\lambda,\mu)_\flat$ are elements in $\mathcal{ML}_{\mathbb{N}}$ whose realizations are the resulting curve systems, respectively; see also Figure 4. We collect in the next lemma some basic properties of these operations, whose proof is left for the reader.

Lemma 2.3. *For $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}$, we have:*

(1) $(\lambda, \mu)_\sharp = (\mu, \lambda)_\flat$.

(2) $(\lambda, \mu)_\sharp = (\lambda, \mu)_\flat$ *if and only if* $i(\lambda, \mu) = 0$. *If* $i(\lambda, \mu) = 0$ *then* $(\lambda, \mu)_\sharp = (\lambda, \mu)_\flat = \lambda + \mu$.

(3) Suppose that every component of μ *intersects* λ *essentially. Then* $((\lambda, \mu)_\sharp, \mu)_\flat = ((\lambda, \mu)_\flat, \mu)_\sharp = \lambda$.

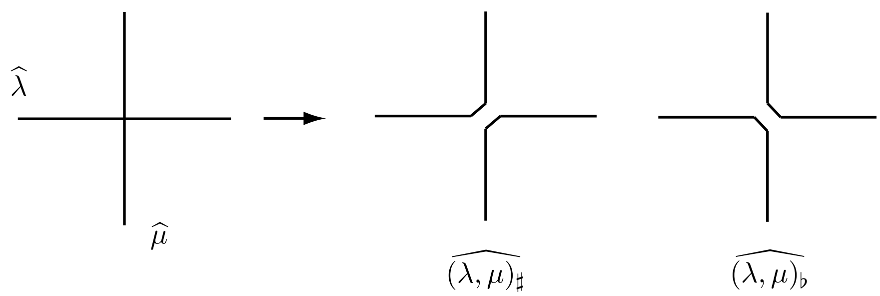

Figure 3: Rules to obtain $(\lambda, \mu)_\sharp$ and $(\lambda, \mu)_\flat$.

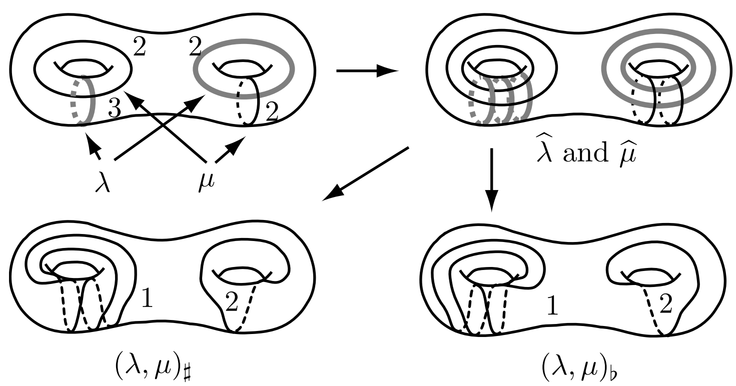

Figure 4: Examples of $(\lambda, \mu)_\sharp$ and $(\lambda, \mu)_\flat$.

2.6 Geometric limits of Kleinian groups

We begin with the definition of Hausdorff convergence.

Definition 2.4 (Hausdorff convergence, geometric convergence). Let X be a locally compact Hausdorff space. A sequence of closed subsets $A_n \subset X$ is said to converge in X to a closed subset $A \subset X$ *in the sense of Hausdorff* if every element

$x \in A$ is the limit of a sequence $\{x_n \in A_n\}$ and if every accumulation point of every sequence $\{x_n \in A_n\}$ lies in A. This is also denoted by $A_n \xrightarrow{\mathrm{H}} A$ in X. A sequence of Kleinian groups Γ_n is said to converge *geometrically* to a group $\widehat{\Gamma}$ if Γ_n converges in $\mathrm{PSL}_2(\mathbb{C})$ to $\widehat{\Gamma}$ in the sense of Hausdorff.

It is a result of Jørgensen and Marden [JM] that if $\rho_n \to \rho_\infty$ in $AH(S)$ then the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$ up to taking a subsequence. Furthermore, the geometric limit $\widehat{\Gamma}$ always contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. The following theorem is due to Kerckhoff and Thurston [KT].

Theorem 2.5 (Kerckhoff-Thurston). *Suppose that a sequence $\rho_n \in \mathcal{QF}$ converges to some $\rho_\infty \in AH(S)$ and that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to $\widehat{\Gamma}$. Then the sequence $\Lambda(\Gamma_n)$ converges in $\widehat{\mathbb{C}}$ to $\Lambda(\widehat{\Gamma})$ in the sense of Hausdorff.*

2.7 Pullbacks of limit sets of Kleinian groups

Let $Y \in P(S)$ be a projective structure with discrete faithful holonomy $\rho_Y : \pi_1(S) \to \Gamma$. Let $\pi_Y : \widetilde{Y} \to Y$ be the universal cover and let $f_Y : \widetilde{Y} \to \widehat{\mathbb{C}}$ be the developing map. Then the preimage $f_Y^{-1}(\Lambda(\Gamma))$ of the limit set $\Lambda(\Gamma)$ in $\widetilde{Y}$ is invariant under the action of the covering transformation group $\pi_1(Y)$. Thus the subset $f_Y^{-1}(\Lambda(\Gamma))$ in $\widetilde{Y}$ descends to the subset

$$\Lambda_Y := \pi_Y \circ f_Y^{-1}(\Lambda(\Gamma))$$

in Y, which is called the *pullback* of the limit set $\Lambda(\Gamma)$ in Y. Similarly, we also obtain the pullback $\pi_Y \circ f_Y^{-1}(\Lambda(\widehat{\Gamma}))$ of the limit set $\Lambda(\widehat{\Gamma})$ for any Kleinian group $\widehat{\Gamma}$ containing Γ. By definition of grafting and Goldman's grafting theorem, an element $Y \in \mathcal{Q}_\lambda$ is characterized as follows:

Lemma 2.6. *Let Y be an element of $Q(S)$. Then $Y \in \mathcal{Q}_\lambda$ if and only if $\Lambda_Y \subset Y$ is a realization of 2λ.*

Suppose that $Y_n \to Y_\infty$ in $P(S)$ as $n \to \infty$. A projective structure on S induces a complex structure on S, and hence a hyperbolic structure. Therefore, with these canonical hyperbolic structures on Y_n and Y_∞, there exist K_n-quasi-isometric maps $\omega_n : Y_\infty \to Y_n$ with $K_n \to 1$ as $n \to \infty$.

Lemma 2.7 (Lemma 3.3 in [It1]). *Suppose that a sequence $Y_n \in Q(S)$ converges to some $Y_\infty \in \overline{Q(S)}$ as $n \to \infty$. Then $\rho_{Y_n} \to \rho_{Y_\infty}$ in $R(S)$. We further assume that the sequence $\Gamma_n = \rho_{Y_n}(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$. Now let $\omega_n : Y_\infty \to Y_n$ be a K_n-quasi-isometric map with $K_n \to 1$ as $n \to \infty$. Then the sequence $\omega_n^{-1}(\Lambda_{Y_n})$ converges in Y_∞ to $\widehat{\Lambda}_{Y_\infty}$ in the sense of Hausdorff, where $\widehat{\Lambda}_{Y_\infty} = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\widehat{\Gamma}))$ is the pullback of the limit set $\Lambda(\widehat{\Gamma})$ in Y_∞. If there is no confusion, we simply say that Λ_{Y_n} converge in Y_∞ to $\widehat{\Lambda}_{Y_\infty}$.*

3 Proof of the main theorem

We devote this section to the proof of Theorem 1.3. Throughout this proof, Figure 5 should be helpful for the reader to understand the arguments. Here we outline the proof. Let ρ_n, Y_n and Z_n be sequences as in introduction. We will recall in §3.1–3.2 constructions and basic facts of sequences ρ_n and Y_n; see [Mc] and [It1] for more details. Suppose that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$. To show that $\{Z_n\}_{n\gg 0} \subset \mathcal{Q}_{(\lambda,\mu)_\sharp}$ and $\{Z_n\}_{n\ll 0} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$, by Lemma 2.6, it suffices to show that the pullback Λ_{Z_n} of $\Lambda(\Gamma_n)$ in Z_n is a realization of $2(\lambda,\mu)_\sharp$ for all $n \gg 0$ and of $2(\lambda,\mu)_\flat$ for all $n \ll 0$. Note that $\Lambda_{Z_n} \subset Z_n$ converge to the pullback $\widehat{\Lambda}_{Z_\infty} \subset Z_\infty$ of $\Lambda(\widehat{\Gamma})$ in the sense of Lemma 2.7. Thus to understand the shape of Λ_{Z_n} we first study in §3.4 the shape of $\widehat{\Lambda}_{Z_\infty} \subset Z_\infty$ by using the result for the shape of the pullback $\widehat{\Lambda}_{Y_\infty} \subset Y_\infty$ of $\Lambda(\widehat{\Gamma})$ obtained in [It1] (see §3.3). We will see in §3.5 that Λ_{Z_n} are obtained by modifying $\widehat{\Lambda}_{Z_\infty}$ at some points which are pullbacks of rank-two parabolic fixed points in $\Lambda(\widehat{\Gamma})$. The reason for the difference of the resulting curve systems Λ_{Z_n} for $n \gg 0$ and for $n \ll 0$ is that the limit sets $\Lambda(\Gamma_n)$ for $n \gg 0$ and for $n \ll 0$ are spiraling in opposite directions at each rank-two parabolic fixed point in $\Lambda(\widehat{\Gamma})$, which will be studied closely in §3.5.

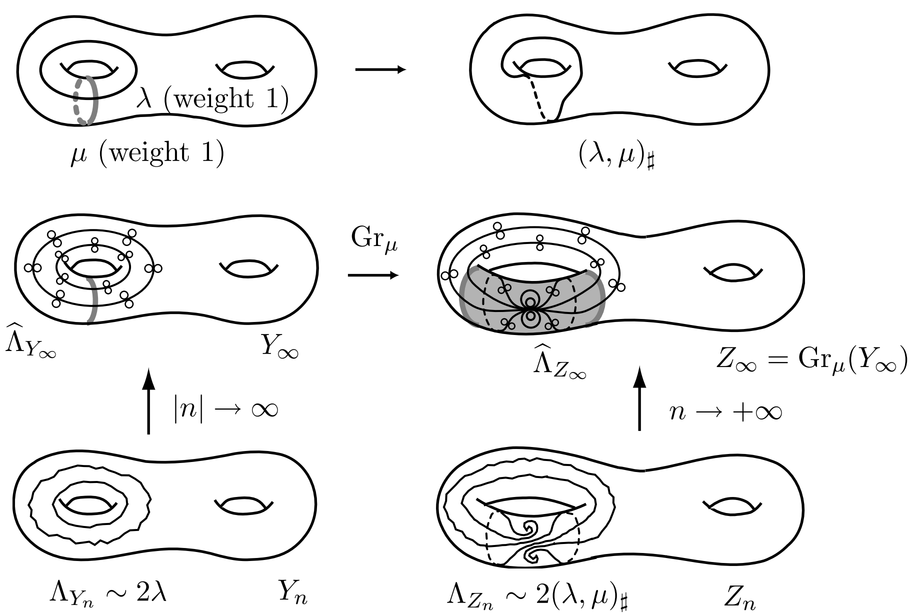

Figure 5: Schematic figure explaining the proof of Theorem 1.3.

3.1 A Kleinian group with rank-two parabolic subgroups

Throughout this section, we fix a non-zero element $\lambda = \sum_{i=1}^{l} k_i c_i \in \mathcal{ML}_{\mathbb{N}}$ with its support $\underline{\lambda} = \sqcup_{i=1}^{l} c_i$. Let

$$M_\lambda = S \times [-1, 1] - \underline{\lambda} \times \{0\}$$

be a 3-manifold $S \times [-1, 1]$ with simple closed curves $c_i \times \{0\}$ $(1 \leq i \leq l)$ removed. Let $\widehat{\Gamma}$ be a Kleinian group whose Kleinian manifold $N_{\widehat{\Gamma}} = (\mathbb{H}^3 \cup \Omega(\widehat{\Gamma}))/\widehat{\Gamma}$ is homeomorphic to M_λ. In what follows, we identify $N_{\widehat{\Gamma}}$ with M_λ via this homeomorphism. Each tubular neighborhood of $c_i \times \{0\}$ in M_λ corresponds to a rank-two cusp end in $N_{\widehat{\Gamma}}$, and hence to a conjugacy class of maximal rank-two parabolic subgroup $\langle \gamma_i, \delta_i \rangle$ in $\widehat{\Gamma} \cong \pi_1(N_{\widehat{\Gamma}})$. We fix the generators of the group $\langle \gamma_i, \delta_i \rangle$ so that $\gamma_i \in \widehat{\Gamma}$ is freely homotopic to $c_i \times \{-1\}$ in $N_{\widehat{\Gamma}}$ and that $\delta_i \in \widehat{\Gamma}$ is trivial in $S \times [-1, 1]$. Moreover, we orient γ_i and δ_i so that for some $\phi_i \in \mathrm{PSL}_2(\mathbb{C})$, $\phi_i \circ \gamma_i \circ \phi_i^{-1}(z) = z + 1$ and $\phi_i \circ \delta_i \circ \phi_i^{-1}(z) = z + \tau_i$ with $\Im \tau_i > 0$.

Note that each connected component ω of $\Omega(\widehat{\Gamma})$ covers a connected component of the conformal boundary $\partial N_{\widehat{\Gamma}} = S \times \{\pm 1\}$ via the quotient map $\mathbb{H}^3 \cup \Omega(\widehat{\Gamma}) \to N_{\widehat{\Gamma}}$, and that the subgroup Γ of $\widehat{\Gamma}$ stabilizing ω is a b-group with $\omega = \Omega_0(\Gamma)$.

3.2 Wrapping maps and associated representations

We introduce here the wrapping map $w_\lambda : S \to N_{\widehat{\Gamma}}$ associated to $\lambda \in \mathcal{ML}_{\mathbb{N}}$, which is an immersion determined up to homotopy. For $0 \in \mathcal{ML}_{\mathbb{N}}$, we let $w_0 : S \to S \times \{-1/2\} \subset N_{\widehat{\Gamma}}$ be the canonical inclusion. The wrapping map $w_\lambda : S \to N_{\widehat{\Gamma}}$ associated to $\lambda \in \sum_{i=1}^{l} k_i c_i \in \mathcal{ML}_{\mathbb{N}}$ is an immersion such that the image $w_\lambda(S)$ in $N_{\widehat{\Gamma}}$ is obtained by cutting $S \times \{-1/2\}$ along $c_i \times \{-1/2\}$ and inserting an annulus which wraps k_i-times around $c_i \times \{0\}$ in $N_{\widehat{\Gamma}}$ at the cut locus for every $1 \leq i \leq l$; see Figure 6. It is also required that w_λ is homotopic to w_0 in $S \times [-1, 1]$. Here we

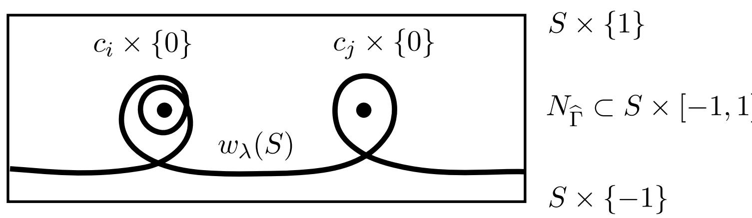

Figure 6: Schematic figure of a wrapping map $w_\lambda : S \to N_{\widehat{\Gamma}}$.

make use of Thurston's Dehn filling theorem ([Co], see also [BO]). By performing simultaneous $(-1, n)$ Dehn filling $(n \in \mathbb{Z})$ on every cusp end of $N_{\widehat{\Gamma}}$, we obtain a sequence of representations $\{\chi_n : \widehat{\Gamma} \to \mathrm{PSL}_2(\mathbb{C})\}$ which satisfies the following:

- $\Gamma_n = \chi_n(\widehat{\Gamma})$ is a quasifuchsian group,

- The kernel of χ_n is normally generated by $\gamma_1^n\delta_1^{-1}, \dots, \gamma_l^n\delta_l^{-1}$,
- χ_n converge algebraically to the identity map of $\widehat{\Gamma}$ as $|n| \to \infty$, and
- Γ_n converge geometrically to $\widehat{\Gamma}$ as $|n| \to \infty$.

Then $\chi_n(\gamma_i) \to \gamma_i$ and $\chi_n(\gamma_i)^n = \chi_n(\gamma_i{}^n) = \chi_n(\delta_i) \to \delta_i$ as $|n| \to \infty$ for each $1 \le i \le l$. Now set

$$\begin{aligned} \rho_n &= \chi_n \circ (w_\lambda)_* &&: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C}), \\ \rho_\infty &= (w_\lambda)_* &&: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C}), \end{aligned}$$

where $(w_\lambda)_* : \pi_1(S) \to \pi_1(N_{\widehat{\Gamma}}) = \widehat{\Gamma}$ is the group isomorphism induced by w_λ. Then ρ_n are faithful representations onto the quasifuchsian groups Γ_n and the sequence ρ_n converges algebraically to ρ_∞. The algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$ is a proper subgroup of the geometric limit $\widehat{\Gamma}$, which is a geometrically finite b-group whose Kleinian manifold N_{Γ_∞} is homeomorphic to $S \times [-1, 1] - \underline{\lambda} \times \{1\}$. Let

$$Y_\infty = \Omega_0(\Gamma_\infty)/\Gamma_\infty$$

be the projective structure on the conformal end of N_{Γ_∞} corresponding to $S \times \{-1\}$. Then $Y_\infty \in \partial\mathcal{Q}_0$ and $hol(Y_\infty) = \rho_\infty$. Since the map hol is a local homeomorphism, there is a sequence $\{Y_n\}_{|n|\gg 0}$ in $Q(S)$ which satisfies $Y_n \to Y_\infty$ as $|n| \to \infty$ and $hol(Y_n) = \rho_n$ for all $|n| \gg 0$. Then it is known by McMullen [Mc] that Y_n are exotic for all $|n| \gg 0$; see Theorem 1.1.

3.3 The pullback of the limit set in Y_∞

Let $\pi_{Y_\infty} : \widetilde{Y}_\infty = \Omega_0(\Gamma_\infty) \to Y_\infty$ be the universal cover. We recall from [It1] the shape of the pullback

$$\widehat{\Lambda}_{Y_\infty} = \pi_{Y_\infty} \circ f_{Y_\infty}^{-1}(\Lambda(\widehat{\Gamma})) = (\Lambda(\widehat{\Gamma}) \cap \Omega_0(\Gamma_\infty))/\Gamma_\infty$$

of the limit set $\Lambda(\widehat{\Gamma})$ in Y_∞. We first fix our terminology: A *regular neighborhood* $\mathcal{N}(\xi)$ of a finite union of (not necessarily disjoint) simple closed curves ξ on S is a 2-dimensional submanifold of S with a deformation retraction $r : \mathcal{N}(\xi) \to \xi$.

Proposition 3.1 (Lemma 4.1 in [It1]). *There is a regular neighborhood $\mathcal{N}(\widehat{\lambda})$ of a realization $\widehat{\lambda}$ of λ in Y_∞ which satisfy the following:*

(1) $\widehat{\Lambda}_{Y_\infty}$ is contained in the interior of $\mathcal{N}(\widehat{\lambda})$, and

(2) for each connected component $\mathcal{N}$ of $\mathcal{N}(\widehat{\lambda})$, $\mathcal{N} \cap \widehat{\Lambda}_{Y_\infty}$ consists of two essential connected component.

Since Λ_{Y_n} converge in Y_∞ to $\widehat{\Lambda}_{Y_\infty}$ in the sense of Lemma 2.7, we can deduce from Proposition 3.1 that Λ_{Y_n} are realization of 2λ, and hence from Lemma 2.6 that $Y_n \in \mathcal{Q}_\lambda$ for all large $|n|$, which implies Theorem 1.2; see [It1] for more details.

In what follows, we will also make use of a regular neighborhood $\mathcal{N}(\underline{\lambda})$ of the support $\underline{\lambda}$ of λ which contains $\mathcal{N}(\widehat{\lambda})$, where a connected component $\mathcal{N}(c_i)$ of $\mathcal{N}(\underline{\lambda})$ contains k_i-parallel components of $\mathcal{N}(\widehat{\lambda})$. Throughout this proof, we fix these regular neighborhoods:

$$\widehat{\Lambda}_{Y_\infty} \subset \mathcal{N}(\widehat{\lambda}) \subset \mathcal{N}(\underline{\lambda}) \subset Y_\infty.$$

Note that Proposition 3.1 (1) implies that $\Lambda(\widehat{\Gamma}) \cap \Omega_0(\Gamma_\infty)$ lies in $\pi_{Y_\infty}^{-1}(\mathcal{N}(\widehat{\lambda}))$, and hence in $\pi_{Y_\infty}^{-1}(\mathcal{N}(\underline{\lambda}))$.

3.4 The pullback of the limit set in Z_∞

Let μ be a non-zero element of $\mathcal{ML}_{\mathbb{N}}$ which has no parallel component in common with λ. Then μ is admissible on Y_∞, and thus the grafting $Z_\infty = \mathrm{Gr}_\mu(Y_\infty)$ of Y_∞ along μ is obtained. We remark that Z_∞ lies in $\partial\mathcal{Q}_\mu$ because the map $\mathrm{Gr}_\mu : \mathcal{Q}_0 \to \mathcal{Q}_\mu$ extends continuously to some neighborhood of $Y_\infty \in \partial\mathcal{Q}_0$; see [Ita]. Let $\pi_{Z_\infty} : \widetilde{Z}_\infty \to Z_\infty$ be the universal cover. We now study the shape of the pullback

$$\widehat{\Lambda}_{Z_\infty} = \pi_{Z_\infty} \circ f_{Z_\infty}^{-1}(\Lambda(\widehat{\Gamma}))$$

of $\Lambda(\widehat{\Gamma})$ in Z_∞ by using the observation of the shape of $\widehat{\Lambda}_{Y_\infty} \subset Y_\infty$ in §3.3.

Proposition 3.2. *There is a regular neighborhood $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ of $\widehat{\lambda} \cup \widehat{\mu}$ in Z_∞ which contains $\widehat{\Lambda}_{Z_\infty}$, where $\widehat{\lambda}$, $\widehat{\mu}$ are realizations of λ, μ whose geometric intersection number is minimal.*

We devote this subsection to the proof of this proposition. For simplicity, we will show in the case where μ is a simple closed curve $d \in \mathcal{S}$ of weight one. Recall that $\pi_{Y_\infty} : \widetilde{Y}_\infty = \Omega_0(\Gamma_\infty) \to Y_\infty$ is the universal cover. Since $d \subset Y_\infty$ is admissible, a connected component $\widetilde{d}$ of $\pi_{Y_\infty}^{-1}(d)$ in $\Omega_0(\Gamma_\infty)$ is $\langle\eta\rangle$-invariant for some loxodromic element $\eta \in \Gamma_\infty$. Let

$$T = (\widehat{\mathbb{C}} - \mathrm{fix}\,(\eta))/\langle\eta\rangle$$

be the quotient torus and let

$$\pi_T : \widehat{\mathbb{C}} - \mathrm{fix}\,(\eta) \to T$$

be the covering map. The simple closed curve $\pi_T(\widetilde{d})$ in T is also denoted by d. We set the longitude l of T as a unique simple closed curve on T (up to homotopy) which is contractible in $\mathbb{H}^3/\langle\eta\rangle$, while d is the meridian of T.

Recall that the grafting $Z_\infty = \mathrm{Gr}_d(Y_\infty)$ is obtained by cutting Y_∞ along d and inserting a cylinder $T - d$:

$$Z_\infty = (Y_\infty - d) \sqcup (T - d).$$

Then the developing map $f_{Z_\infty} : \widetilde{Z}_\infty \to \widehat{\mathbb{C}}$ is obtained by gluing piece by piece the developing maps

$$f_{Y_\infty} : \widetilde{Y}_\infty = \Omega_0(\Gamma_\infty) \to \widehat{\mathbb{C}}, \quad f_T : \widetilde{T} = \widehat{\mathbb{C}} - \mathrm{fix}\,(\eta) \to \widehat{\mathbb{C}}$$

and their conjugations by elements of Γ_∞. (Note that both f_{Y_∞} and f_T are equal to the identity map.) Therefore the set $\widehat{\Lambda}_{Z_\infty} \cap (Y_\infty - d)$ is equal to $\widehat{\Lambda}_{Y_\infty} \setminus d$ and is contained in $\mathcal{N}(\widehat{\lambda}) \setminus d$ as a subset of $Y_\infty - d \subset Z_\infty$, and the set $\widehat{\Lambda}_{Z_\infty} \cap (T - d)$ is equal to $\widehat{\Lambda}_T \setminus d$ as a subset of $T - d \subset Z_\infty$, where

$$\widehat{\Lambda}_T = \pi_T \circ f_T^{-1}(\Lambda(\widehat{\Gamma})) = \pi_T(\Lambda(\widehat{\Gamma}) - \mathrm{fix}\,(\eta))$$

is the pullback of $\Lambda(\widehat{\Gamma})$ in T. Thus we have

$$\widehat{\Lambda}_{Z_\infty} = (\widehat{\Lambda}_{Y_\infty} \setminus d) \sqcup (\widehat{\Lambda}_T \setminus d).$$

We first consider the case of $i(\lambda, d) = 0$. We may assume that $\mathcal{N}(\widehat{\lambda}) \cap d = \emptyset$. Then we have $\widehat{\Lambda}_{Y_\infty} \cap d = \emptyset$ in Y_∞ from $\widehat{\Lambda}_{Y_\infty} \subset \mathcal{N}(\widehat{\lambda})$. Moreover, this implies that $\Lambda(\widehat{\Gamma}) \cap \widetilde{d} = \emptyset$ in $\widehat{\mathbb{C}}$, and hence that $\widehat{\Lambda}_T \cap d = \emptyset$ in T. Thus we obtain $\widehat{\Lambda}_{Z_\infty} \subset \mathcal{N}(\widehat{\lambda} \cup d)$ by setting a regular neighborhood $\mathcal{N}(\widehat{\lambda} \cup d)$ of $\widehat{\lambda} \cup d$ as

$$\mathcal{N}(\widehat{\lambda} \cup d) = \mathcal{N}(\widehat{\lambda}) \sqcup \mathcal{N}_T,$$

where $\mathcal{N}_T$ is an annulus in $T - d \subset Z_\infty$ which contains $\widehat{\Lambda}_T$.

From now on, we assume that $i(\lambda, \mu) = i(\lambda, d) \neq 0$ and set $s = i(\underline{\lambda}, d)$. Here we fix our terminology:

Definition 3.3 (crescent). A closed set $A \subset \widehat{\mathbb{C}}$ is called a *crescent* if A is the closure of $B_2 - B_1$, where B_1, B_2 are topological closed discs in $\widehat{\mathbb{C}}$ such that $B_1 \subset B_2$ and that $\partial B_1 \cap \partial B_2$ consists of one point p. We say that A is touching at p. A closed subset in a Riemann surface homeomorphic to a crescent is also called a crescent.

We first consider the shape of $\widehat{\Lambda}_T$ in T:

Lemma 3.4. *There exist crescents $\{\mathcal{A}_j\}_{j=1}^s$ and closed balls $\{\mathcal{B}_j\}_{j=1}^s$ in T which satisfy the following (see Figure 7):*

(1) $\mathcal{A}_1, \cdots, \mathcal{A}_s$ are mutually disjoint. Each $\mathcal{A}_j$ is homotopically equivalent to the longitude l in T. We let $p_j \in T$ denote the touching point of $\mathcal{A}_j$.

(2) Interiors of $\mathcal{B}_1, \cdots, \mathcal{B}_s$ are mutually disjoint.

(3) The set $\bigcup_{j=1}^s \mathcal{B}_j$ is connected and homotopically equivalent to the meridian d in T. In addition, it is satisfied that $\mathcal{A}_j \cap \left(\bigcup_{j=1}^s \mathcal{B}_j\right) = \{p_j\}$ for each $1 \leq j \leq s$, which implies that $\bigcup_{j=1}^s \mathcal{B}_j$ is a string of beads.

(4) $\widehat{\Lambda}_T \subset \left(\bigcup_{j=1}^s \mathcal{A}_j\right) \cup \left(\bigcup_{j=1}^s \mathcal{B}_j\right)$.

Proof. We begin with studying the subset $\pi_T(\Lambda(\widehat{\Gamma}) \cap \Omega_0(\Gamma_\infty))$ of $\widehat{\Lambda}_T$. Recall from Proposition 3.1 that the set $\Lambda(\widehat{\Gamma}) \cap \Omega_0(\Gamma_\infty)$ is contained in the preimage $\pi_{Y_\infty}^{-1}(\mathcal{N}(\underline{\lambda}))$ of $\mathcal{N}(\underline{\lambda})$ in $\Omega_0(\Gamma_\infty)$. Let $\widetilde{A}$ be a connected component of $\pi_{Y_\infty}^{-1}(\mathcal{N}(\underline{\lambda}))$ and let $\gamma \in \Gamma_\infty$

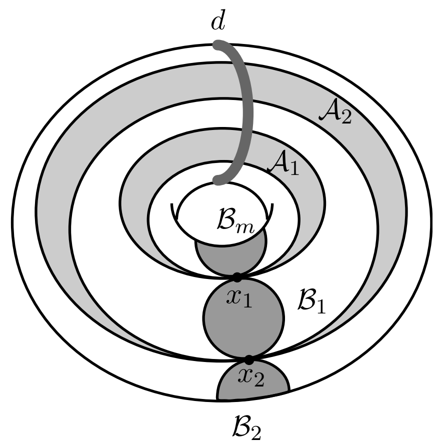

Figure 7: Crescents $\{\mathcal{A}_j\}_{j=1}^s$ and balls $\{\mathcal{B}_j\}_{j=1}^s$ in T.

be a parabolic element such that $\gamma(\widetilde{A}) = \widetilde{A}$. Then $\widetilde{A} \cup \operatorname{fix}(\gamma)$ is a crescent, which mapped injectively by π_T into T. If $\widetilde{A} \cap \widetilde{d} = \emptyset$, then $\widetilde{A} \cup \operatorname{fix}(\gamma)$ does not separate two fixed points of η and $\pi_T(\widetilde{A} \cup \operatorname{fix}(\gamma))$ is contractible in T. On the other hand, if $\widetilde{A} \cap \widetilde{d} \neq \emptyset$, then $\widetilde{A} \cup \operatorname{fix}(\gamma)$ separates two fixed points of η and $\pi_T(\widetilde{A} \cup \operatorname{fix}(\gamma))$ is homotopically equivalent to the longitude l in T. Recall that $s = i(\underline{\lambda}, d)$ is the intersection number of $\underline{\lambda}$ and d in Y_∞. Then there exist exactly s-crescents

$$\mathcal{A}_1, \dots, \mathcal{A}_s$$

in T each of which is a component of $\pi_T \circ \pi_{Y_\infty}^{-1}(\mathcal{N}(\underline{\lambda}))$ with its touching point $p_j \in \mathcal{A}_j$ and is homotopically equivalent to the longitude l. Since any two components of $\underline{\lambda}$ are not parallel in Y_∞, we see that $p_i \neq p_j$ for $1 \leq i \neq j \leq s$, and hence that $\mathcal{A}_1, \cdots, \mathcal{A}_s$ are mutually disjoint. Note that the intersection $\widehat{\Lambda}_T \cap d$ lies in $\bigcup_{j=1}^s \mathcal{A}_j$ from the argument above. We now divide $\widehat{\Lambda}_T$ into

$$\widehat{\Lambda}_T = \widehat{\Lambda}_T^1 \sqcup \widehat{\Lambda}_T^2,$$

where $\widehat{\Lambda}_T^1 = \widehat{\Lambda}_T \cap \bigcup_{j=1}^s \mathcal{A}_j$ and $\widehat{\Lambda}_T^2 = \widehat{\Lambda}_T \setminus \bigcup_{j=1}^s \mathcal{A}_j$. Then $\widehat{\Lambda}_T^2$ is contained in the complement of $d \cup \bigcup_{j=1}^s \mathcal{A}_j$ in T, which consist of s-open balls. Moreover, since $\partial\mathcal{A}_j - \{p_j\}$ does not intersect $\widehat{\Lambda}_T$ for each j, one see that the intersection of the closure of $\widehat{\Lambda}_T^2$ with $\widehat{\Lambda}_T^1$ is $\{p_1, \dots, p_s\}$. Therefore, by a slight modification of those open balls, we obtain a string of beads $\bigcup_{j=1}^s \mathcal{B}_j$ which contains $\widehat{\Lambda}_T^2$ and which satisfies the desired conditions. □

Observe that for each $1 \leq j \leq s$ the crescent $\mathcal{A}_j$ contains l_j-crescents

$$\mathcal{A}_j^{(1)}, \dots, \mathcal{A}_j^{(l_j)}$$

which are components of $\pi_T \circ \pi_{Y_\infty}^{-1}(\mathcal{N}(\widehat{\lambda}))$ with common touching point p_j. Here l_j is the weight of the component of λ associated to $\mathcal{A}_j$. Then it is also satisfied the

statement of Lemma 3.5 with $\mathcal{A}_j$ replaced by $\bigcup_{m=1}^{l_j} \mathcal{A}_j^{(m)}$ for all j. Adding suitable (mutually disjoint) closed neighborhoods V_j of p_j, we obtain a submanifold

$$\mathcal{N}_T = \left(\bigcup_{j=1}^{s} V_j \right) \cup \left(\bigcup_{j=1}^{s} \bigcup_{m=1}^{l_j} \mathcal{A}_j^{(m)} \right) \cup \left(\bigcup_{j=1}^{s} \mathcal{B}_j \right)$$

in T which contains $\widehat{\Lambda}_T$, and a regular neighborhood

$$\mathcal{N}(\widehat{\lambda} \cup d) = (\mathcal{N}(\widehat{\lambda}) \setminus d) \sqcup (\mathcal{N}_T \setminus d)$$

of $\widehat{\lambda} \cup d$ in Z_∞ which contains $\widehat{\Lambda}_{Z_\infty} = (\widehat{\Lambda}_{Y_\infty} \setminus d) \sqcup (\widehat{\Lambda}_T \setminus d)$. Thus we obtain the result of Proposition 3.3 in the case of $\mu = d \in \mathcal{S}$. The result for general $\mu \in \mathcal{ML}_{\mathbb{N}}$ is also obtained by the same argument.

3.5 Cutting $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ into $\mathcal{N}(\widehat{(\lambda,\mu)_\sharp})$ and $\mathcal{N}(\widehat{(\lambda,\mu)_\flat})$

Since Λ_{Z_n} converge in Z_∞ to $\widehat{\Lambda}_{Z_\infty}$ in the sense of Lemma 2.7 and since $\widehat{\Lambda}_{Z_\infty}$ lies in $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ from Proposition 3.2, it follows that Λ_{Z_n} lie in $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ for all large $|n|$. Given large enough $n > 0$, we will show that there is a system of simple arcs which cuts $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ into a regular neighborhood $\mathcal{N}(\widehat{(\lambda,\mu)_\sharp})$ of a realization of $(\lambda,\mu)_\sharp$ containing Λ_{Z_n}; see Figure 8. The same argument yields $\mathcal{N}(\widehat{(\lambda,\mu)_\flat})$ which contains Λ_{Z_n} for given $n < 0$ with large enough $|n|$.

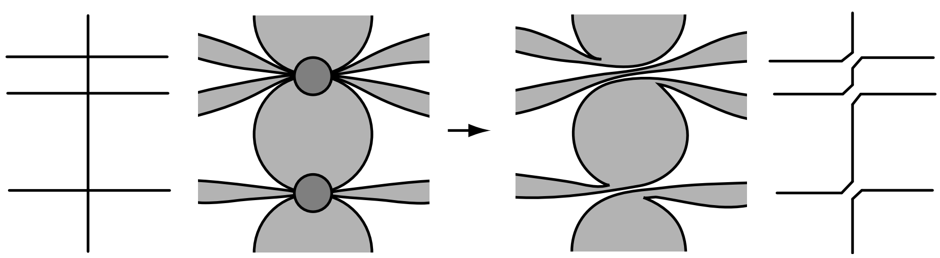

Figure 8: Cutting $\mathcal{N}(\widehat{\lambda} \cup \widehat{\mu})$ into $\mathcal{N}(\widehat{(\lambda,\mu)_\sharp})$. The graphs in both-sides are illustrating $\widehat{\lambda} \cup \widehat{\mu}$ and $\widehat{(\lambda,\mu)_\sharp}$ respectively.

To describe more precisely, we again concentrate our attention to the case of $\mu = d \in \mathcal{S}$. Let $\mathcal{N}(\widehat{\lambda} \cup d)$ be a regular neighborhood of $\widehat{\lambda} \cup d$ containing $\widehat{\Lambda}_{Z_\infty}$ obtained as in §3.4. Note that for each j the point $p_j \in \widehat{\Lambda}_{Z_\infty}$ is the pullback of a common fixed point $p \in \Lambda(\widehat{\Gamma})$ of a maximal rank-two parabolic subgroup $\langle \gamma, \delta \rangle$ of $\widehat{\Gamma}$. We suppose that the generators γ, δ satisfy the convention as in §3.1. Let $U_j \subset Z_\infty$ be an open neighborhood of p_j containing V_j and let $\phi_j : U_j \to U$ be a homeomorphism from $U_j \subset Z_\infty$ to an open neighborhood $U \subset \widehat{\mathbb{C}}$ of p. We set $V = \phi_j(V_j)$. Let ω be a connected component of $\Omega(\widehat{\Gamma})$ which contains p in its boundary. Then ω is simply connected and is invariant under the action of γ. We

may assume that $\omega \cap U$ consists of exactly two components ω_1 and ω_2 for which $\gamma(\omega_1) \subset \omega_1$ and $\gamma^{-1}(\omega_2) \subset \omega_2$; see Figure 9. In addition, we assume that $\delta(\omega_2)$ and $\delta^{-1}(\omega_2)$ are contained in U. In this situation, we have the following:

Proposition 3.5. *Let $x \in \omega_1$, $y \in \omega_2$ such that x, y, $\delta(y), \delta^{-1}(y) \notin V$. Then there exist simple arcs β_n which satisfy the following:*

(1) $\beta_n \subset \Omega(\Gamma_n)$ for all $|n| \gg 0$,

(2) β_n joins x and $\delta(y)$ and is contained in $\omega_1 \cup V \cup \delta(\omega_2)$ for all $n \gg 0$, and

(3) β_n joins x and $\delta^{-1}(y)$ and is contained in $\omega_1 \cup V \cup \delta^{-1}(\omega_2)$ for all $n \ll 0$.

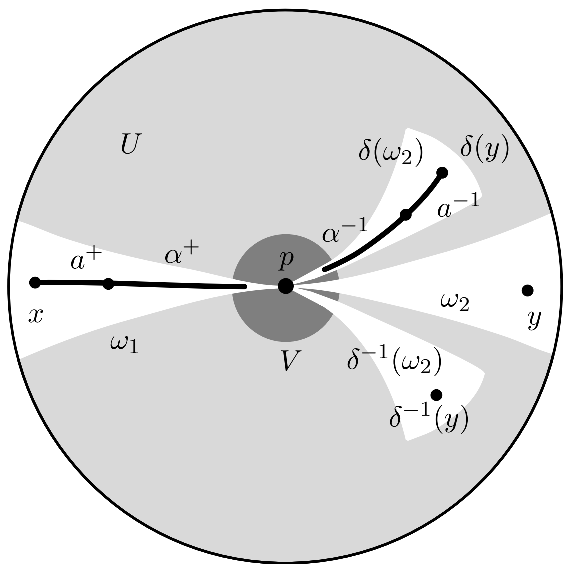

Figure 9: Figure explaining Proposition 3.7 and its proof.

Before proving the proposition above, We shall complete the proof of Theorem 1.3. We fix $1 \leq j \leq s$ for a wile. Recall that $p_j \in U_j$ is the touching point of l_j-crescents $A_j^{(1)}, \dots, A_j^{(l_j)}$ in T. Thus we may assume that $U_j \setminus \mathcal{N}(\widehat{\lambda} \cup d)$ consists of exactly $2l_j + 2$ connected components. We first consider the case of $n \gg 0$. Given large enough $n > 0$, it follows from Proposition 3.5 that there are mutually disjoint l_j-simple arcs

$$\beta_n^{(1)}, \dots, \beta_n^{(l_j)}$$

in U_j which satisfy the following:

- $\beta_n^{(m)}$ does not intersect $\widehat{\Lambda}_{Z_\infty}$ for each $1 \leq m \leq l_j$,
- the intersection $\beta_n^{(m)} \cap \mathcal{N}(\widehat{\lambda} \cup d)$ is contained in V_j for each $1 \leq m \leq l_j$, and

- all $2l_j$-end points of $\beta_n^{(1)}, \dots, \beta_n^{(l_j)}$ are contained in distinct components of $U_j \setminus \mathcal{N}(\widehat{\lambda} \cup d)$.

Observe that by cutting $\mathcal{N}(\widehat{\lambda} \cup d)$ along those arcs contained in U_j for every $1 \leq j \leq s$, we obtain a regular neighborhood $\mathcal{N}(\widehat{(\lambda, d)_\sharp})$ of a realization of $(\lambda, d)_\sharp$ which contains Λ_{Z_n}; see Figure 8. Note that since Z_n lies in $Q(S)$, Lemma 2.6 implies that Λ_{Z_n} consists of mutually disjoint non-trivial simple closed curves contained in $\mathcal{N}(\widehat{(\lambda, d)_\sharp})$. We claim that each connected component of $\mathcal{N}(\widehat{(\lambda, d)_\sharp})$ contains exactly two connected components of Λ_{Z_n}, and hence that Λ_{Z_n} is a realization of $2(\lambda, d)_\sharp$. In fact, the claim holds because each connected component of $\mathcal{N}(\widehat{(\lambda, d)_\sharp})$ contains a connected component of $\mathcal{N}(\widehat{\lambda}) \setminus d$ which contains exactly two essential connected components of $\widehat{\Lambda}_{Y_\infty} \setminus d$; see [It1] for more rigorous argument. Thus Theorem 2.6 implies that $Z_n \in \mathcal{Q}_{(\lambda,d)_\sharp}$. Similarly, we have $Z_n \in \mathcal{Q}_{(\lambda,d)_\flat}$ for all $n \ll 0$. Therefore we have completed the proof of Theorem 1.3 for the case of $\mu = d \in \mathcal{S}$. The result for general $\mu \in \mathcal{ML}_{\mathbb{N}}$ is obtained by the same argument. All what we have to do is to show Proposition 3.5.

Proof of Proposition 3.5. We only consider the case of $n \gg 0$, since the argument for the case $n \ll 0$ is completely parallel. Recall that there is a sequence $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \gamma$ and $(\gamma_n)^n \to \delta$ as $n \to +\infty$. Assume for simplicity that y lies in the $\langle\gamma\rangle$-orbit of x; i.e. $y = \gamma^{-n_0}(x)$ for some positive integer n_0. Then $\delta(y) = \gamma^{-n_0}\delta(x)$ holds since γ commutes with δ. Let $N > 0$ be a positive integer such that for all $k \geq N$, $\gamma^k(x)$ and $\gamma^{-k}\delta(y)$ are contained in V. Let a^+ be an arc in ω_1 joining x and $\gamma(x)$. Since $\Omega(\Gamma_n)$ converge to $\Omega(\widehat{\Gamma})$ in the sense of Carathéodory (see [KT]), $a^+ \subset \Omega(\widehat{\Gamma})$ are contained in $\Omega(\Gamma_n)$ for all $n \gg 0$. Hence there exist arcs b_n^+ joining x and $\gamma_n(x)$ in $\Omega(\Gamma_n)$ which satisfy $b_n^+ \xrightarrow{\mathrm{H}} a^+$ as $n \to +\infty$. Now set

$$\beta_n = \bigcup_{k=0}^{n-n_0-1} (\gamma_n)^k(b_n^+).$$

Then β_n is an arc which joins x to $(\gamma_n)^{n-n_0}(x)$. Since $(\gamma_n)^{n-n_0}(x)$ converge to $\gamma^{-n_0}\delta(x) = \delta(y)$ as $n \to +\infty$, we may assume that the arc β_n joins x to $\delta(y)$. We will show that β_n satisfy the desired conditions. Since $b_n^+ \subset \Omega(\Gamma_n)$ and $\gamma_n \in \Gamma_n$, it follows that $\beta_n \subset \Omega(\Gamma_n)$. Let us now consider two subarcs

$$\beta_n^+ = \bigcup_{k=0}^{N} (\gamma_n)^k(b_n^+), \quad \beta_n^- = \bigcup_{k=n-n_0-1-N}^{n-n_0-1} (\gamma_n)^k(b_n^+)$$

of β_n, which are the unions of the first N-orbits and the last N-orbits, respectively. Then we will obtain $\beta_n \subset \omega_1 \cup V \cup \delta(\omega_2)$ by showing that $\beta_n^+ \subset \omega_1$, $\beta_n^- \subset \delta(\omega_2)$ and $\beta_n - (\beta_n^+ \cup \beta_n^-) \subset V$ for all $n \gg 0$.

Claim 1: $\beta_n^+ \subset \omega_1$ and $\beta_n^- \subset \delta(\omega_2)$ for all $n \gg 0$.

Let a^- denote the arc $\gamma^{-n_0-1}\delta(a^+)$ joining $\gamma^{-1}\delta(y)$ and $\delta(y)$. We may assume that $a^- \subset \delta(\omega_2)$. Let us consider two arcs

$$\alpha^+ = \bigcup_{k=0}^{N} \gamma^k(a^+), \quad \alpha^- = \bigcup_{k=0}^{N} \gamma^{-k}(a^-),$$

which are contained in ω_1 and $\delta(\omega_2)$, respectively. To obtain the claim, it suffices to show that $\beta_n^+ \xrightarrow{\mathrm{H}} \alpha^+$ and $\beta_n^- \xrightarrow{\mathrm{H}} \alpha^-$ in $\widehat{\mathbb{C}}$. Since $\gamma_n \to \gamma$ and $b_n^+ \xrightarrow{\mathrm{H}} a^+$, it follows that $\beta_n^+ \xrightarrow{\mathrm{H}} \alpha^+$. Now set $b_n^- = (\gamma_n)^{n-n_0-1}(b_n^+)$. Then we have $b_n^- \xrightarrow{\mathrm{H}} a^- = \gamma^{-n_0-1}\delta(a^+)$ from $(\gamma_n)^{n-n_0-1} \to \gamma^{-n_0-1}\delta$ and $b_n^+ \xrightarrow{\mathrm{H}} a^+$. Thus we conclude that

$$\beta_n^- = \bigcup_{l=0}^{N}(\gamma_n)^{-l}(\gamma_n)^{n-n_0-1}(b_n^+) = \bigcup_{l=0}^{N}(\gamma_n)^{-l}(b_n^-) \ \xrightarrow{\mathrm{H}}\ \bigcup_{l=0}^{N}(\gamma_n)^{-l}(a^-) = \alpha^-.$$

Claim 2: $\beta_n - (\beta_n^+ \cup \beta_n^-) \subset V$ for all $n \gg 0$.

We will show that the orbits $\{(\gamma_n)^k(x) \,|\, N < k < n - n_0 - 1 - N\}$ of x in $\beta_n - (\beta_n^+ \cup \beta_n^-)$ are contained in V. The claim is obtained by the same argument. Since V is a neighborhood of the common fixed point p of $\langle \gamma, \delta \rangle$, we may assume that by changing N large enough if necessary, $|s| \leq N$ holds whenever $\gamma^s\delta^t(x)$ is not contained in the interior $\mathrm{int}\,(V)$ of V. To obtain a contradiction, suppose that there is a sequence $k_n \to \infty\,(n \to \infty)$ such that $N < k_n < n - n_0 - 1 - N$ and that $(\gamma_n)^{k_n}(x) \notin V$. Then passing to a subsequence if necessary, $(\gamma_n)^{k_n}(x)$ converges to some point $\hat{x} \notin \mathrm{int}\,(V)$. Then the sequence $(\gamma_n)^{k_n}$ does not diverge in $\mathrm{PSL}_2(\mathbb{C})$ because both fixed points of γ_n converge to p and $(\gamma_n)^{k_n}(x)$ converge to $\hat{x} \neq p$. Since $\langle \gamma_n \rangle \xrightarrow{\mathrm{H}} \langle \gamma, \delta \rangle$ in $\mathrm{PSL}_2(\mathbb{C})$, after passing to a further subsequence, we may assume that $(\gamma_n)^{k_n} \to \gamma^s\delta^t$ for some $s,\, t \in \mathbb{Z}$. Then it follows that $\hat{x} = \gamma^s\delta^t(x) \notin \mathrm{int}\,(V)$, and hence that $|s| \leq N$. On the other hand, since the convergence $(\gamma_n)^{k_n} \to \gamma^s\delta^t$ implies the convergence $(\gamma_n)^{k_n - tn - s} \to id$ as $n \to +\infty$, we have $k_n \equiv tn + s\,(n \gg 0)$ from Lemma 3.6 in [JM]. It then follows from $N < k_n < n - n_0 - 1 - N$ that $t = 0$, $s \geq 0$ or $t = 1$, $s \leq 0$, both of which contradict to $|s| \leq N$. ☐

4 Corollaries of the main theorem

We collect here some consequences of Theorem 1.3.

Theorem 4.1. *For any non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, there exists $Y \in \overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_\lambda}$ such that $U \cap \mathcal{Q}_\lambda$ is disconnected for any sufficiently small neighborhood U of Y. In particular, $\overline{\mathcal{Q}_\lambda}$ is not a topological manifold with boundary.*

Proof. The result follows directly from Theorem 1.3 in the case of $i(\lambda, \mu) \neq 0$. ☐

We will show in Theorem 4.3 that any two components of $Q(S)$ bump from Theorem 1.3 combined with the following result, which is a generalization of Theorem 1.2.

Theorem 4.2 (Theorem B in [It1]). *Let $\{\lambda_i\}_{i=1}^m$ be a finite subset of $\mathcal{ML}_{\mathbb{N}} - \{0\}$ such that $i(\lambda_i, \lambda_j) = 0$ for any $1 \le i < j \le m$. Then we have $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{\lambda_1}} \cap \cdots \cap \overline{\mathcal{Q}_{\lambda_m}} \neq \emptyset$.*

Theorem 4.3. *For any $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}$, we have $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$.*

Proof. Let λ, $\mu \in \mathcal{ML}_{\mathbb{N}}$. If $i(\lambda, \mu) = 0$, then the result follows from Theorem 4.2. From now on, we assume that $i(\lambda, \mu) \neq 0$ and decompose μ into $\mu = \mu' + \mu''$ so that μ', $\mu'' \in \mathcal{ML}_{\mathbb{N}}$ and $i(\lambda, \mu) = i(\lambda, \mu')$ are satisfied. The proof is divided into two cases (i) $\mu'' = 0$ and (ii) $\mu'' \neq 0$.

(i) Case of $\mu'' = 0$. Then $\mu = \mu'$. By the same argument as in §3, replacing λ to $(\lambda, \mu)_\flat$, we obtain a sequence

$$\{Y_n\}_{|n| \gg 0} \subset \mathcal{Q}_{(\lambda,\mu)_\flat}$$

which converges to some $Y_\infty \in \partial \mathcal{Q}_0$ as $|n| \to \infty$. Since $(\lambda, \mu)_\flat$ and μ have no parallel component in common, we obtain a grafting $\mathrm{Gr}_\mu(Y_\infty) \in \partial \mathcal{Q}_\mu$ of Y_∞ and a convergent sequence

$$Z_n \to \mathrm{Gr}_\mu(Y_\infty) \quad (|n| \to \infty)$$

which satisfies $hol(Z_n) = hol(Y_n)$ for all large $|n|$. It follows from Theorem 1.3 that $Z_n \in \mathcal{Q}_{((\lambda,\mu)_\flat,\mu)_\sharp} = \mathcal{Q}_\lambda$ for all $n \gg 0$. Thus we obtain $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$.

(ii) Case of $\mu'' \neq 0$. Since $i(\lambda, \mu'') = 0$ and $i(\mu', \mu'') = 0$, we have $i((\lambda, \mu')_\flat, \mu'') = 0$. Theorem 4.2 then implies that $\overline{\mathcal{Q}_0} \cap \overline{\mathcal{Q}_{(\lambda,\mu')_\flat}} \cap \overline{\mathcal{Q}_{\mu''}} \neq \emptyset$. More precisely, we obtain as in §3 two sequences

$$\{Y_n\}_{|n| \gg 0} \subset \mathcal{Q}_{(\lambda,\mu')_\flat}, \quad \{Y'_n\}_{|n| \gg 0} \subset \mathcal{Q}_{\mu''}$$

both of which converge to some $Y_\infty \in \partial \mathcal{Q}_0$ as $|n| \to \infty$; see the proof of Theorem B in [It1] for more details. Note that the holonomy image of Y_∞ is a b-group whose parabolic locus is the support of $(\lambda, \mu')_\flat + \mu''$. Since $(\lambda, \mu')_\flat + \mu''$ and μ' have no parallel component in common, we obtain a grafting $\mathrm{Gr}_{\mu'}(Y_\infty)$ and two convergent sequences

$$Z_n,\, Z'_n \to \mathrm{Gr}_{\mu'}(Y_\infty) \quad (|n| \to \infty)$$

which satisfy $hol(Z_n) = hol(Y_n)$ and $hol(Z'_n) = hol(Y'_n)$ for all large $|n|$. It follows from Theorem 1.3 that $Z_n \in \mathcal{Q}_{((\lambda,\mu')_\flat,\mu')_\sharp} = \mathcal{Q}_\lambda$ and $Z'_n \in \mathcal{Q}_{\mu'+\mu''} = \mathcal{Q}_\mu$ for all $n \gg 0$. Thus we obtain $\overline{\mathcal{Q}_\lambda} \cap \overline{\mathcal{Q}_\mu} \neq \emptyset$ also in this case. □

Theorem 4.4. *For any non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the holonomy map $hol : P(S) \to R(S)$ is not injective on $\overline{\mathcal{Q}_\lambda}$, although it is invective on $\mathcal{Q}_\lambda$.*

Proof. Let ν be an element of $\mathcal{ML}_{\mathbb{N}}$ each of whose component intersects λ essentially. Lemma 2.3 then implies that

$$(\nu, (\lambda,\nu)_\sharp)_\sharp = ((\lambda,\nu)_\sharp, \nu)_\flat = \lambda,$$
$$(\nu, (\lambda,\nu)_\flat)_\flat = ((\lambda,\nu)_\flat, \nu)_\sharp = \lambda.$$

Now let $\{Y_n\}_{|n|\gg 0} \subset Q_\nu$ be a sequence constructed as in §3 which converges to some $Y_\infty \in \partial\mathcal{Q}_0$ as $|n| \to \infty$. Applying Theorem 1.3, we obtain two sequences in $\mathcal{Q}_\lambda$ as follows:

- $\{Z_n\}_{n\gg 0}$ in $\mathcal{Q}_{(\nu,(\lambda,\nu)_\sharp)_\sharp} = \mathcal{Q}_\lambda$ which converges to $\mathrm{Gr}_{(\lambda,\nu)_\sharp}(Y_\infty)$ as $n \to +\infty$.
- $\{Z'_n\}_{n\ll 0}$ in $\mathcal{Q}_{(\nu,(\lambda,\nu)_\flat)_\flat} = \mathcal{Q}_\lambda$ which converges to $\mathrm{Gr}_{(\lambda,\nu)_\flat}(Y_\infty)$ as $n \to -\infty$.

Since $i(\lambda,\nu) \neq 0$, we have $(\lambda,\nu)_\sharp \neq (\lambda,\nu)_\flat$. Therefore $\mathrm{Gr}_{(\lambda,\nu)_\sharp}(Y_\infty) \neq \mathrm{Gr}_{(\lambda,\nu)_\flat}(Y_\infty)$ and the result follows. □

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